6 PASKAITA
Iracionaliųjų funkcijų integravimas. Trigonometrinių funkcijų integravimas

Integruojant iracionaliąsias funkcijas reikia parinkti tokį keitinį, kad iracionalioji funkcija virstų racionaliąja.

1. $\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right)dx$ keitinys yra $x = t^k$, kur $k -$ trupmenų $\frac{m}{n}, \ldots, \frac{r}{s}$ bendrasis vardiklis.

1) $\int \frac{\sqrt{x}}{\sqrt[3]{x^2} - \sqrt[6]{x}}dx = \begin{bmatrix} x = t^6, t = \sqrt[6]{x} \\ dx = 6t^5 dt \end{bmatrix} = \int \frac{t^3 6t^5 dt}{t^4 - t} = 6\int \frac{t^7}{t^3 - 1}dt = 6\int \left(t^4 + t + \frac{t}{t^3 - 1}\right)dt = 6\int t^4 dt + 6\int t dt + 6\int \frac{t}{t^3 - 1}dt =$

$= 6\frac{t^5}{5} + 6\frac{t^2}{2} + 6\int \frac{t}{(t-1)(t^2 + t + 1)}dt = \ldots = 6\frac{t^5}{5} + 3t^2 + 2\ln|t - 1| - \ln|t^2 + t + 1| + 2\sqrt{3}arctg\frac{2t + 1}{\sqrt{3}} + C =$

$\frac{6}{5}x^{5/6} + 3x^{1/3} + 2\ln\left|x^{1/6} - 1\right| - \ln\left|x^{1/3} + x^{1/6} + 1\right| + 2\sqrt{3}arctg\frac{2x^{1/6} + 1}{\sqrt{3}} + C$

2) $\int \frac{dx}{\sqrt[3]{x} + \sqrt{x}} = 6\left(\frac{\sqrt{x}}{3} - \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} - \ln\left(1 + \sqrt[6]{x}\right)\right) + C$;

2. $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)dx$ keitinys yra $\frac{ax+b}{cx+d} = t^k$, $x = \frac{dt^k - b}{a - ct^k}$, $k -$ bendras mažiausias n ir s kartotinis

3) $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x} = \begin{bmatrix} \frac{1-x}{1+x} = t^2 \\ x = \frac{1-t^2}{1+t^2} \\ dx = -\frac{4tdt}{(1+t^2)^2} \end{bmatrix} = 4\int \frac{t^2 dt}{(t^2-1)(t^2+1)} = 2\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + 2arctgt + C = \ln\left|\frac{\sqrt{\frac{1-x}{1+x}} - 1}{\sqrt{\frac{1-x}{1+x}} + 1}\right| + 2arctg\sqrt{\frac{1-x}{1+x}} + C$

4) $\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}dx = \begin{bmatrix} \frac{1+x}{x} = t^2;\ x = \frac{1}{t^2 - 1} \\ dx = -\frac{2tdt}{(t^2-1)^2} \end{bmatrix} = \int \frac{(t^2-1)^2 t(-2t)dt}{(t^2-1)^2} = -2\frac{t^3}{3} + C = -\frac{2}{3}\left(\frac{1+x}{x}\right)^{3/2} + C$;

Integruoti trigonometrinėms funkcijoms taikomas kintamųjų pakeitimo metodas.
$\int R(\sin x, \cos x)$ čia $R -$ kintamųjų $\sin x$ ir $\cos x$ racionalioji funkcija.

Parenkame universalų keitinį $t = tg\frac{x}{2}$, tai $x = 2arctgt;\ dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{tg\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{2t}{1+t^2}$; $\cos x = \frac{1-t^2}{1+t^2}$.

1) $\int \frac{dx}{5 - 4\sin x + 3\cos x} = \begin{bmatrix} t = tg\frac{x}{2};\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2dt}{1+t^2};\ \sin x = \frac{2t}{1+t^2} \end{bmatrix} = \int \frac{2}{1+t^2}\frac{dt}{5 - \frac{8t}{1+t^2} + \frac{3(1-t^2)}{1+t^2}} = \frac{1}{2 - tg\frac{x}{2}} + C$;

2) $\int \frac{dx}{5 + 4\cos x} = \left[t = tg\frac{x}{2}\right] = \int \frac{2dt}{1+t^2}\frac{1}{5 + 4\left(\frac{1-t^2}{1+t^2}\right)} = 2\int \frac{dt}{t^2 + 9} = 2\frac{1}{3}arctg\frac{t}{3} + C = \frac{2}{3}arctg\frac{tg\frac{x}{2}}{3} + C$

Tačiau tokį keitinį nevisada tikslinga taikyti. Kartais racionaliau pasirinkti paprastesnius keitinius.

(1) Jei pointegralinė funkcija $R(\sin x, \cos x)$ yra nelyginė $\sin x$ atžvilgiu, tada tinka keitinys $t = \cos x$, $x = \arccos t$, $dx = -\frac{dt}{\sqrt{1-t^2}}$, $\sin x = \sqrt{1-t^2}$

(2) Jei pointegralinė funkcija $R(\sin x,\cos x)$ yra nelyginė $\cos x$ atžvilgiu, tada tinka keitinys

$$t=\sin x,\quad x=\arcsin t,\quad dx=\frac{dt}{\sqrt{1-t^2}},\quad \cos x=\sqrt{1-t^2}$$

(3) Jei pointegralinė funkcija $R(\sin x,\cos x)$ yra lyginė $\sin x$ ir $\cos x$ atžvilgiu, tada tinka keitinys

$$t=tgx,\quad x=\text{arc}\,tgt,\quad dx=\frac{dt}{1+t^2},\quad \sin x=\frac{t}{\sqrt{1+t^2}},\cos x=\frac{1}{\sqrt{1+t^2}}$$

3) $$\int\frac{\sin xdx}{(1-\cos x)^2}=\begin{bmatrix} t=\cos x,\ dx=-\dfrac{dt}{\sqrt{1-t^2}}\\ \sin x=\sqrt{1-t^2}\end{bmatrix}=-\int\frac{\sqrt{1-t^2}}{(1-t)^2}\frac{dt}{\sqrt{1-t^2}}=-\int\frac{dt}{(1-t)^2}=\int\frac{d(1-t)}{(1-t)^2}=-\frac{1}{1-t}+C=\frac{1}{t-1}+C=\frac{1}{\cos x-1}+C;$$

4) $$\int\frac{\cos^3 x}{1+\sin x}dx=\begin{bmatrix} t=\sin x,\ dx=\dfrac{dt}{\sqrt{1-t^2}}\\ \cos x=\sqrt{1-t^2}\end{bmatrix}=\int\frac{(1-t^2)^{3/2}}{1+t}\frac{dt}{(1-t^2)^{1/2}}=\int\frac{1-t^2}{1+t}dt=\int(1-t)dt=t-\frac{t^2}{2}+C=\sin x-\frac{\sin^2 x}{2}+C;$$

5) $$\int\frac{dx}{4-3\cos^2x+5\sin^2x}=\begin{bmatrix} t=tgx,\ dx=\dfrac{dt}{1+t^2}\\ \sin x=\dfrac{t}{\sqrt{1+t^2}},\ \cos x=\dfrac{1}{\sqrt{1+t^2}}\end{bmatrix}=\int\frac{dt}{1+t^2}\frac{1}{4-\frac{3}{1+t^2}+\frac{5t^2}{1+t^2}}=\int\frac{dt}{1+9t^2}=\frac13\int\frac{d(3t)}{1+(3t)^2}=$$

$$=\frac13 arctg(3t)+C=\frac13 arctg(3tgx)+C;$$

Kai kurie integralai $\int R(\sin x,\cos x)dx$ nesunkiai apskaičiuojami panaudojant trigonometrines formules:

$$\int\sin^m x\cos^n xdx,$$

Jei $m=2k+1>0,$ tai $t=\cos x,\ dt=-\sin xdx$

Jei $n=2k+1>0,$ tai $t=\sin x,\ dt=\cos xdx$

6) $$\int\frac{\sin^3x}{\cos^4x}dx=\begin{bmatrix} t=\cos x\\ dt=-\sin xdx\\ \sin^2x=1-\cos^2x\end{bmatrix}=-\int\frac{(1-t^2)}{t^4}dt=\int\frac{t^2}{t^4}dt-\int\frac{dt}{t^4}=-\frac1t+\frac{1}{3t^3}+C;$$

Jei $m+n=-2k$ ar $m+n=0$, tai $t=tgx$.

7) $$\int\frac{dx}{\sin^3x\cos^3x}=\begin{bmatrix} t=tgx,\ dx=\dfrac{dt}{1+t^2}\\ \sin x=\dfrac{t}{\sqrt{1+t^2}},\ \cos x=\dfrac{1}{\sqrt{1+t^2}}\end{bmatrix}=\int\frac{1}{\frac{t^3}{(1+t^2)^{3/2}}\frac{1}{(1+t^2)^{3/2}}}\frac{dt}{(1+t^2)}=\int\frac{(1+t^2)^2}{t^3}dt=\int\frac{1+2t^2+t^4}{t^3}dt=$$

$$=\int\frac{dt}{t^3}+2\int\frac{dt}{t}+\int tdt=-\frac{1}{2t^2}+2\ln|t|+\frac{t^2}{2}+C=-\frac{1}{2tg^2x}+2\ln|tgx|+\frac{tg^2x}{2}+C;$$

Jei turim $\int tg^l xdx$:

kai $l>0,\quad t=tgx$

kai $l<0,\quad t=ctgx,\quad dx=-\frac{dt}{1+t^2}$

8) $$\int\frac{dx}{tg^5x}=\int ctg^5xdx=\begin{bmatrix} t=ctgx\\ dx=-\dfrac{dt}{1+t^2}\end{bmatrix}=-\int\frac{t^5dt}{1+t^2}=-\int\left(t^3-t+\frac{t}{1+t^2}\right)dt=-\int t^3dt+\int tdt-\frac12\int\frac{d(1+t^2)}{1+t^2}=-\frac{t^4}{4}+\frac{t^2}{2}-\frac12\ln|1+t^2|+C=$$

$$==-\frac{ctg^4x}{4}+\frac{ctg^2x}{2}-\frac12\ln|1+ctg^2x|+C;$$

Jei $m=2k>0$ ir $n=2k>0$, tai panaudojam formules

$$\sin^2x=\frac{1-\cos2x}{2},\cos^2x=\frac{1+\cos2x}{2}.$$

9) $$\int\sin^2x\cos^4x=\int\sin^2x\cos^2x\frac{1+\cos2x}{2}dx=\frac12\frac14\int\sin^22x(1+\cos2x)dx=\frac18\int\sin^22xdx+\frac18\int\sin^22x\cos2xdx=$$

$$=\frac18\int\frac{(1-\cos4x)}{2}dx+\frac12\int\frac12\sin^22xd(\sin2x)=\frac1{16}\int dx-\frac1{16}\int\cos4x\frac14d(4x)+\frac1{16}\frac{\sin^32x}{3}=\frac{x}{16}-\frac{1}{64}\sin4x+\frac{1}{48}\sin^32x+C;$$

Kitos formulės:

(1) $\int \cos(mx)\cos(nx)\,dx = \int \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]dx$

(2) $\int \sin(mx)\sin(nx)\,dx = \int \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]dx$

(3) $\int \sin(mx)\cos(nx)\,dx = \int \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]dx$

10) $\int \sin 7x \cos 5x dx = \frac{1}{2}\int\left(\sin(7+5)x + \sin(7-5)x\right)dx = \frac{1}{2}\int \sin 12x + \frac{1}{2}\int \sin 2x dx = -\frac{1}{2}\frac{1}{12}\cos 12x - \frac{1}{2}\frac{1}{2}\cos 2x + C$;

11) $\int \frac{dx}{(1+\cos x)\sin x} = \begin{bmatrix} t = tg\frac{x}{2},\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2} \end{bmatrix} = \int \frac{2dt}{(1+t^2)} \frac{1}{\left(1+\frac{1-t^2}{1+t^2}\right)\frac{2t}{1+t^2}} = \int \frac{(1+t^2)}{2t}dt = \frac{1}{2}\int\frac{dt}{t} + \frac{1}{2}\int t dt = \frac{1}{2}\ln|t| + \frac{1}{4}t^2 + C =$

$= \frac{1}{2}\ln\left|tg\frac{x}{2}\right| + \frac{1}{4}tg^2\frac{x}{2} + C =$

Apskaičiuokite integralus naudodami kintamojo keitimo metodą:

1. $\int \frac{dx}{1+\sqrt{x}}$	2. $\int \frac{dx}{\sqrt[3]{x}\left(\sqrt[3]{x}-1\right)}$

3. $\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}dx$	4. $\int \frac{dx}{1+\sqrt{x+1}}$	5. $\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}dx$	6. $\int \frac{\sqrt{x}}{1+x}dx$
7. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}$	8. $\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[8]{x}}dx$	9. $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$	10. $\int \frac{dx}{x\left(\sqrt{x}+\sqrt[5]{x^2}\right)}$
11. $\int \frac{dx}{x\sqrt{x+1}}$	12. $\int \frac{dx}{1+\sqrt[3]{x+1}}$		

1. $\int \frac{dx}{5-4\sin x+3\cos x}$	2. $\int \frac{dx}{4+5\sin^2 x-3\cos^2 x}$	3. $\int \frac{1-\sin x}{\cos x}dx$	4. $\int \frac{\sin^3 x}{\cos x}dx$
5. $\int \frac{\sin x dx}{(1-\cos x)^2}$	6. $\int \cos x \sin 3x dx$	7. $\int \sin^4 x dx$	8. $\int tg^3 x dx$
9. $\int \frac{dx}{1-\cos x}$	10. $\int \frac{dx}{1+\sin x}$	11. $\int \frac{dx}{5+4\sin x}$	12. $\int \frac{dx}{5-3\cos x}$
13. $\int \frac{1-\cos x}{1+\cos x}dx$	14. $\int \frac{1+\sin x}{1-\sin x}dx$	15. $\int \frac{dx}{\sin x+\cos x}$	16. $\int tg^4 x dx$
17. $\int \frac{\cos 2x}{1+\sin x\cos x}dx$	18. $\int \frac{dx}{1+tgx}$	19. $\int \frac{dx}{1-\sin^4 x}$	20. $\int \frac{dx}{1+\sin^2 x}$
21. $\int \frac{\cos^3 x}{\sin^4 x}dx$	22. $\int \sin^4 x\cos^2 x dx$	23. $\int \sin 2x \sin 5x dx$	24. $\int \frac{dx}{\sin^6 x}$
25. $\int \frac{dx}{\cos^4 x}$	26. $\int \cos^2 x \sin^3 x dx$	27. $\int \frac{\sin^4 x}{\cos^2 x}dx$	28. $\int \frac{dx}{\cos x\cdot\sin^3 x}$